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An approximate analytic method for solving 1D dual-phase-lagging heat transport equations

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Abstract

In this study, we develop an approximate analytic method for solving 1D dual-phase-lagging heat conduction equations, which are derived based on the original dual-phase-lagging model without the first-order Taylor series approximation. The approximate analytic solution is obtained by employing the method of separation of variables. The coefficients of the series solution are then approximated by polynomials. The numerical method is illustrated with two simple examples. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Heat transport through thin films is of vital importance in microtechnology applications [1,2]. Thin films of metals, of dielectrics such as SiO_2 or Si semiconductors are important components of microelectronic devices. Reducing the device size to microscale enhances the switching speed of the device. Size reduction, however, increases the rate of heat generation which leads to a high thermal load on the microdevice. Heat transfer at the microscale is also important for the processing of materials with a pulsed-laser [3,4]. Examples in metal processing are laser micromachining, laser patterning, laser processing of diamond films from carbon ion implanted copper substrates, and laser surface hardening. Hence, studying the thermal behavior of thin films is essential for predicting the performance of a microelectronic device or for obtaining the desired microstructure [2]. A dual-phase-lagging model has been used for studying the lagging response in conductive heat transfer at the microscale. The lagging response describes the heat flux vector and the temperature gradient occurring at different instants of time in the heat transfer process. The original dual-phase-lagging heat transport equations are expressed as [5]:

$$-\nabla \cdot \vec{q} + Q = \rho C_{\mathrm{p}} \frac{\partial T}{\partial t}, \tag{1}$$

$$\vec{q}(x, y, z, t + \tau_{\mathrm{q}}) = -k \nabla T(x, y, z, t + \tau_{\mathrm{T}}), \tag{2}$$

where $\vec{q} = (q_1, q_2, q_3)$ is heat flux , T is temperature, k is conductivity, C_{p} is specific heat, ρ is density, Q is a heat source, τ_{q} and τ_{T} are positive constants, which are the time lags of the heat flux and temperature gradient, respectively. In the classical theory of diffusion, the heat flux vector ($\vec{q}$) and the temperature gradient (∇T) across a material volume are assumed to occur at the same instant of time. They satisfy the Fourier's law of heat conduction

$$\vec{q}(x, y, z, t) = -k \nabla T(x, y, z, t). \tag{3}$$

As Tzou [5] pointed out, the thermal lagging describes the fast-transient effect of thermal inertia. The finite time required for the energy exchange/thermal activation in microscale resides in the phase lag of the temperature gradient. It also describes the microstructural interaction effect in space in terms of the resulting delayed response in time.Using Taylor series expansion, the first-order approximation of Eq. (2) gives [5]

$$\vec{q} + \tau_{\mathrm{q}} \frac{\partial \vec{q}}{\partial t} = -k \left[\nabla T + \tau_{\mathrm{T}} \frac{\partial}{\partial t} [\nabla T] \right]. \tag{4}$$

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Nomenclature

a	constant $= (k/\rho C_{\rm p})$
C_i	coefficient in Taylor series of $\Gamma(t)$
$C_{\rm p}$	specific heat
k	conductivity
l	length of the considered interval
M, n	integers
Q	heat source
$\vec{q} = (q_1, q_2, q_3)$	heat flux
T	temperature
t	time
x, y, z	Cartesian coordinates
Greek symbols	
β	constant $= (-a\lambda_n)$
Γ	coefficient in Fourier series of $T(x,t)$
λ_n	constant $= ((n\pi/l)^2)$
Φ	coefficient in Fourier series of the source term $S(x,t)$
ρ	density
$\tau_{\rm q}$	time lag of heat flux
$\tau_{\rm T}$	time lag of temperature gradient

Analytic and numerical methods for solving the above coupled Eqs. (1) and (4) have been widely studied [5–21]. Among these, Tzou and Ozisik [5,6] considered Eqs. (1) and (4) in 1D and eliminated the heat flux $\vec{q}$ to obtain a heat transport equation as follows:

$$A\frac{\partial T}{\partial t} + D\frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + B\frac{\partial^3 T}{\partial x^2 \partial t} + S, \tag{5}$$

where $A = (\rho C_{\rm p})/k$, $B = \tau_{\rm T}$, $D = (\tau_{\rm q}\rho C_{\rm p})/k$, and $S = 1/k(Q + \tau_{\rm q}(\partial Q)/(\partial t))$. They studied the lagging behavior by solving the above heat transport Eq. (5) without body heating in a semiinfinite interval, $[0, +\infty)$. The solution was obtained by using the Laplace transform method and the Riemann-sum approximation for the inversion [8]. Wang et al. [9,10] developed methods of measuring the phase lags of the heat flux and the temperature gradient and obtained analytical solutions for 1D, 2D and 3D heat conduction domains under essentially arbitrary initial and boundary conditions. The solution structure theorems were also developed for both mixed and Cauchy problems of dual-phase-lagging heat conduction equations. Tang and Araki [12] derived the analytic solution in finite rigid slabs by using the Green's function method and a finite integral transform technique. Lin et al. [13] obtained the analytic solution using the Fourier series. Al-Nimr and Arpaci [14] proposed a new approach, based on the physical decoupling of the hyperbolic two-step model, to describe the thermal behavior of a thin metal film exposed to picoseconds thermal pulses. Chen and Beraun [15] employed the corrective smoothed particle method to obtain a numerical solution of ultrashort laser pulse interactions with metal films. Dai and Nassar [16] developed a two-level finite difference scheme of the Crank–Nicholson type by introducing an intermediate function for solving Eq. (5) in a finite interval. It is shown by the discrete energy method that the scheme is unconditionally stable. The scheme has been generalized to a 3D rectangular thin film case where the thickness is at the sub-microscale [17]. Further, Dai and Nassar [18,19] developed high-order compact finite difference schemes for solving Eqs. (5) and (4) in a 3D thin film, respectively. Dai and Nassar also developed several numerical methods for solving the coupled Eqs. (1) and (4) in double-layered thin films [20,21].

Since Eq. (4) is only a first-order Taylor series approximation of Eq. (2), it may not be a good representation of the original dual-phase-lagging equation, Eq. (2). Further, the higher-order Taylor series approximation of Eq. (2) may result in higher-order derivatives, which may be difficult to solve both analytically and numerically. Therefore, it is useful to study thermal behavior based on the original coupled Eqs. (1) and (2). Through this, one can study the high-order effect in $\tau_{\rm q}$ and $\tau_{\rm T}$. In this paper, we will develop an approximate analytic method for solving 1D dual-phase-lagging heat transport equations.

2. Approximate analytic method

We first consider the coupled dual-phase-lagging heat conduction equations (Eqs. (1) and (2)) without the heat source in 1D

$$-\frac{\partial q}{\partial x} = \rho C_{\rm p}\frac{\partial T}{\partial t} \tag{6}$$

and

$$q(x, t+\tau_{\rm q}) = -k\frac{\partial T}{\partial x}(x, t+\tau_{\rm T}). \tag{7}$$

Eliminating q, we obtain a 1D dual-phase-lagging heat conduction equation as follows:

$$\frac{\partial T(x, t+\tau_{\rm q})}{\partial t} = a\frac{\partial^2 T(x, t+\tau_{\rm T})}{\partial x^2}, \tag{8}$$

where $a = k/(\rho C_{\rm p})$. The initial and boundary conditions are assumed to be

$$T(x, 0) = \varphi(x) \tag{9}$$

and

$$T(0,t) = T(l,t) = 0. \tag{10}$$

Assume that the solution of the above problem exists and is smooth. To solve the above problem, we first employ the method of separation of variables. Letting $T(x,t) = \Gamma(t)X(x)$ and substituting it into Eq. (8), we obtain

$$\Gamma'(t+\tau_q)X(x) = a\Gamma(t+\tau_T)X''(x).$$

Separating variables x and t, we have

$$\frac{\Gamma'(t+\tau_q)}{a\Gamma(t+\tau_T)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant. Thus, the problem, Eqs. (8)–(10), can be separated into solving two different ordinary equations as follows:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(l) = 0 \tag{11}$$

and

$$\Gamma'(t+\tau_q) = -a\lambda\Gamma(t+\tau_T). \tag{12}$$

It is readily seen that the solution of Eq. (11) is

$$X_n(x) = \sin\frac{n\pi x}{l}, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \; n = 1,2,3,\ldots \tag{13}$$

Hence, Eq. (12) becomes

$$\Gamma_n'(t+\tau_q) = -a\lambda_n\Gamma_n(t+\tau_T), \qquad \Gamma_n(0) = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}\,dx, \; n = 1,2,\ldots \tag{14}$$

Once Eq. (14) is solved, the analytic solution of the problem, Eqs. (8)–(10), can be written as follows:

$$T(x,t) = \sum_{n=1}^{\infty} \Gamma_n(t)\sin\frac{n\pi x}{l}. \tag{15}$$

The remaining question is how to find the solution of Eq. (14). It is noted that Eq. (14) is difficult to solve analytically, because τ_q and τ_T are different values. To overcome this difficulty, we develop an approximate analytic method, in which $\Gamma_n(t)$ is approximated by a polynomial. Once $\Gamma_n(t)$ is obtained, then the solution $T(x,t)$ is obtained from Eq. (15). To this end, we first rewrite Eq. (14) without the index n for convenience

$$\Gamma'(t+\tau_q) = \beta\Gamma(t+\tau_T), \tag{16}$$

where β is a constant. Let

$$\Gamma(t) = \sum_{i=0}^{M} C_i t^i, \tag{17}$$

where M could be a large integer and $C_0 = \Gamma(0)$. Substituting Eq. (17) into Eq. (16) gives

$$\sum_{i=1}^{M} C_i i(t+\tau_q)^{i-1} = \beta\sum_{i=0}^{M} C_i(t+\tau_T)^i. \tag{18}$$

Case 1. If $\tau_T \geqslant \tau_q$, we let $t = -\tau_q$ in Eq. (18) and obtain

$$C_1 = \beta\sum_{i=0}^{M} C_i(\tau_T - \tau_q)^i.$$

Differentiating Eq. (18) with respect to t and letting $t = -\tau_q$ give

$$\sum_{i=2}^{M} C_i i(i-1)(t+\tau_q)^{i-2} = \beta\sum_{i=1}^{M} C_i i(t+\tau_T)^{i-1} \tag{19}$$

and

$$2!C_2 = \beta\sum_{i=1}^{M} C_i i(\tau_T-\tau_q)^{i-1}.$$

Continuously, differentiating Eq. (19) with respect to t and letting $t = -\tau_q$ give

$$\sum_{i=3}^{M} C_i i(i-1)(i-2)(t+\tau_q)^{i-3} = \beta\sum_{i=2}^{M} C_i i(i-1)(t+\tau_T)^{i-2} \tag{20}$$

and

$$3!C_3 = \beta\sum_{i=2}^{M} C_i i(i-1)(\tau_T-\tau_q)^{i-2}.$$

In general, we have

$$k!C_k = \beta\sum_{i=k-1}^{M} C_i i(i-1)\cdots(i-k+2)(\tau_T-\tau_q)^{i-k+1}, \quad k = 1,\ldots,M. \tag{21}$$

We now solve the coefficients C_k, $k = 1,\ldots,M$, from Eq. (21). Letting $k = M$, we obtain from Eq. (21)

$$M!C_M = \beta[M!(\tau_T-\tau_q)C_M + (M-1)!C_{M-1}].$$

Hence,

$$C_M = \frac{a_{M-1}}{M}C_{M-1}, \quad a_{M-1} = \frac{\beta}{1-\beta(\tau_T-\tau_q)}.$$

Letting $k = M-1$, we obtain

$$\begin{aligned}(M-1)!C_{M-1} &= \beta\sum_{i=M-2}^{M} C_i i(i-1)\cdots(i-M+3)(\tau_T-\tau_q)^{i-M+2} \\ &= \beta\left[\frac{M!}{2!}(\tau_T-\tau_q)^2C_M + (M-1)!(\tau_T-\tau_q)C_{M-1} + (M-2)!C_{M-2}\right] \\ &= \beta\left[\frac{(M-1)!}{2!}a_{M-1}(\tau_T-\tau_q)^2C_{M-1} + (M-1)!(\tau_T-\tau_q)C_{M-1} + (M-2)!C_{M-2}\right].\end{aligned}$$

Hence,

$$C_{M-1} = \frac{a_{M-2}}{M-1} C_{M-2},$$

$$a_{M-2} = \frac{\beta}{1 - \beta(\tau_T - \tau_q) - \beta \frac{a_{M-1}}{2!} (\tau_T - \tau_q)^2}.$$

Again, letting $k = M-2$, we obtain

$$\begin{aligned}(M-2)! C_{M-2} &= \beta \sum_{i=M-3}^{M} C_i i(i-1)\cdots(i-M+4)(\tau_T - \tau_q)^{i-M+3} \\ &= \beta \left[\frac{M!}{3!}(\tau_T - \tau_q)^3 C_M + \frac{(M-1)!}{2!}(\tau_T - \tau_q)^2 C_{M-1} \right. \\ &\quad \left. + (M-2)!(\tau_T - \tau_q) C_{M-2} + (M-3)! C_{M-3} \right].\end{aligned}$$

Replacing C_M by C_{M-1} and then C_{M-1} by C_{M-2} based on the previous equations, we obtain

$$C_{M-2} = \frac{a_{M-3}}{M-2} C_{M-3},$$

where

$$a_{M-3} = \frac{\beta}{1 - \beta(\tau_T - \tau_q) - \beta \frac{a_{M-2}}{2!}(\tau_T - \tau_q)^2 - \beta \frac{a_{M-1} a_{M-2}}{3!} (\tau_T - \tau_q)^3}.$$

In general, we have

$$a_{M-k} = \frac{\beta}{1 - \beta(\tau_T - \tau_q) - \beta \sum_{i=2}^{k} \frac{1}{i!} (\tau_T - \tau_q)^i \prod_{j=1}^{i-1} a_{M-k+j}}, \quad k = 2, \ldots, M \tag{22}$$

and

$$C_{M-k+1} = \frac{a_{M-k}}{M-k+1} C_{M-k}, \quad C_0 = \Gamma(0), \quad k = M, \ldots, 1, \tag{23}$$

where

$$a_{M-1} = \frac{\beta}{1 - \beta(\tau_T - \tau_q)}. \tag{24}$$

Case 2. If $\tau_T < \tau_q$, we let $t = -\tau_T$ in Eq. (18) and obtain

$$\sum_{i=1}^{M} C_i i (\tau_q - \tau_T)^{i-1} = \beta C_0.$$

Differentiating Eq. (18) with respect to t and letting $t = -\tau_T$ give

$$\sum_{i=2}^{M} C_i i(i-1)(t + \tau_q)^{i-2} = \beta \sum_{i=1}^{M} C_i i (t + \tau_T)^{i-1} \tag{25}$$

and

$$\sum_{i=2}^{M} C_i i(i-1)(\tau_q - \tau_T)^{i-2} = \beta C_1.$$

Again, differentiating Eq. (25) with respect to t and letting $t = -\tau_T$ give

$$\sum_{i=3}^{M} C_i i(i-1)(i-2)(t + \tau_q)^{i-3} = \beta \sum_{i=2}^{M} C_i i(i-1)(t + \tau_T)^{i-2}$$

and

$$\sum_{i=3}^{M} C_i i(i-1)(i-2)(t + \tau_q)^{i-3} = 2! \beta C_2.$$

In general, we have

$$\beta k! C_k = \sum_{i=k+1}^{M} C_i i(i-1)\cdots(i-k)(\tau_q - \tau_T)^{i-k-1}, \quad k = 0, \ldots, M-1. \tag{26}$$

We then solve the coefficients C_k, $k = 1, \ldots, M$, from Eq. (26). Letting $k = M-1$, we obtain from Eq. (26)

$$\beta (M-1)! C_{M-1} = M! C_M.$$

Hence,

$$C_M = \frac{a_{M-1}}{M} C_{M-1}, \quad a_{M-1} = \beta.$$

Letting $k = M-2$, we obtain

$$\begin{aligned}\beta(M-2)! C_{M-2} &= \sum_{i=M-1}^{M} C_i i(i-1)\cdots(i-M+2)(\tau_q - \tau_T)^{i-M+1} \\ &= M!(\tau_q - \tau_T) C_M + (M-1)! C_{M-1} \\ &= (M-1)! a_{M-1} (\tau_q - \tau_T) C_{M-1} + (M-1)! C_{M-1}.\end{aligned}$$

Hence,

$$C_{M-1} = \frac{a_{M-2}}{M-1} C_{M-2}, \quad a_{M-2} = \frac{\beta}{1 + a_{M-1}(\tau_q - \tau_T)}.$$

Again, letting $k = M-3$, we obtain

$$\begin{aligned}\beta(M-3)! C_{M-3} &= \sum_{i=M-2}^{M} C_i i(i-1)\cdots(i-M+3)(\tau_q - \tau_T)^{i-M+2} \\ &= \frac{M!}{2!}(\tau_q - \tau_T)^2 C_M + (M-1)!(\tau_q - \tau_T) C_{M-1} \\ &\quad + (M-2)! C_{M-2}.\end{aligned}$$

Replacing C_M by C_{M-1} and then C_{M-1} by C_{M-2} based on the previous equations, we obtain

$$C_{M-2} = \frac{a_{M-3}}{M-2} C_{M-3},$$

where

$$a_{M-3} = \frac{\beta}{1 + a_{M-2}(\tau_q - \tau_T) + \frac{a_{M-1} a_{M-2}}{2!} (\tau_q - \tau_T)^2}.$$

In general, we obtain

$$a_{M-k} = \frac{\beta}{1 + \sum_{i=1}^{k-1} \frac{1}{i!}(\tau_q - \tau_T)^i \prod_{j=1}^{i} a_{M-k+j}}, \quad k = 1, \ldots, M \tag{27}$$

and

$$C_{M-k+1} = \frac{a_{M-k}}{M-k+1} C_{M-k}, \quad C_0 = \Gamma(0), \quad k = M, \ldots, 1. \tag{28}$$

Once C_k is obtained, $\Gamma(t) = \sum_{i=0}^{M} C_i t^i$. Finally, one can obtain $\Gamma_n(t)$ for each n based on the above method (i.e., $\Gamma_n(t) = \sum_{i=0}^{M} C_i^n t^i$), and hence the approximate analytic solution for Eqs. (8)–(10) can be written as follows:

$$T(x,t) = \sum_{n=1}^{\infty} \Gamma_n(t) \sin\frac{n\pi x}{l}, \tag{29}$$

where

$$\Gamma_n(t) = \sum_{i=0}^{M} C_i^n t^i. \tag{30}$$

It should be pointed out that the above approximate solution satisfies exactly the following initial conditions:

$$T(x,0) = \varphi(x), \quad \frac{\partial^k T(x,0)}{\partial t^k} = 0 \ \ (k = M+1, M+2, \ldots)$$

and

$$\text{if } \tau_T \geqslant \tau_q, \quad \frac{\partial^k T(x,0)}{\partial t^k} = \beta \frac{\partial^{k-1} T(x, \tau_T - \tau_q)}{\partial t^{k-1}} \quad (k = 1, 2, \ldots, M)$$

$$\text{if } \tau_T \leqslant \tau_q, \quad \frac{\partial^k T(x, \tau_T - \tau_q)}{\partial t^k} = \beta \frac{\partial^{k-1} T(x,0)}{\partial t^{k-1}} \quad (k = 1, 2, \ldots, M).$$

Further, the convergence in the series, Eq. (29), needs to be further studied mathematically.

We now generalize the above idea to develop an approximate analytic solution for the coupled dual-phase-lagging heat conduction equations (Eqs. (1) and (2)) in 1D, where the heat source $Q(x,t)$ is included

$$-\frac{\partial q(x,t)}{\partial x} + Q(x,t) = \rho C_p \frac{\partial T(x,t)}{\partial t} \tag{31}$$

and

$$q(x, t+\tau_q) = -k \frac{\partial T}{\partial x}(x, t+\tau_T). \tag{32}$$

Assume that ρ, C_p, and k are constants. We eliminate q in Eqs. (31) and (32) to obtain a 1D dual-phase-lagging heat conduction equation with a heat source as follows:

$$\frac{\partial T(x, t+\tau_q)}{\partial t} = a \frac{\partial^2 T(x, t+\tau_T)}{\partial x^2} + S(x, t+\tau_q), \tag{33}$$

where $a = k/(\rho C_p)$ and $S(x,t) = 1/(\rho C_p) Q(x,t)$. The initial and boundary conditions are assumed to be

$$T(x,0) = \varphi(x) \tag{34}$$

and

$$T(0,t) = T(l,t) = 0. \tag{35}$$

We assume that the solution of the above problem exists and is smooth. To solve the above problem, we first employ the method of separation of variables and let

$$T(x,t) = \sum_{n=1}^{\infty} \Gamma_n(t) \sin\frac{n\pi x}{l} \tag{36}$$

and

$$S(x,t) = \sum_{n=1}^{\infty} \Phi_n(t) \sin\frac{n\pi x}{l}. \tag{37}$$

Substituting Eqs. (36) and (37) into Eq. (33) gives

$$\Gamma_n'(t+\tau_q) = -a\lambda_n \Gamma_n(t+\tau_T) + \Phi_n(t+\tau_q), \tag{38}$$

where $\lambda_n = (n\pi/l)^2$ and

$$\Gamma_n(0) = \frac{2}{l} \int_0^l \varphi(x) \sin\frac{n\pi x}{l}\, dx, \quad n = 1, 2, \ldots \tag{39}$$

To find an approximate analytic solution, we first rewrite Eq. (38) without the index n for convenience

$$\Gamma'(t+\tau_q) = \beta \Gamma(t+\tau_T) + \Phi(t+\tau_q), \tag{40}$$

where $\beta = -a\lambda_n$. Let

$$\Gamma(t) = \sum_{i=0}^{M} C_i t^i, \tag{41}$$

where M could be a large integer and $C_0 = \Gamma(0)$. Substituting Eq. (41) into Eq. (40) gives

$$\sum_{i=1}^{M} C_i i (t+\tau_q)^{i-1} = \beta \sum_{i=0}^{M} C_i (t+\tau_T)^i + \Phi(t+\tau_q). \tag{42}$$

Using a similar argument as described in the previous problem, we obtain two results as follows.

Case 1. If $\tau_T \geqslant \tau_q$, then

$$a_{M-k} = (M-k+1)\frac{a_{M-k+1}}{\beta} - M(M-1)\cdots \times (M-k+1)\frac{(\tau_T-\tau_q)^k}{k!} - \sum_{i=1}^{k-1} (M-k+1)\cdots(M-k+i) a_{M-k+i} \frac{(\tau_T-\tau_q)^i}{i!}, \tag{43}$$

$$b_{M-k} = (M-k+1)\frac{b_{M-k+1}}{\beta} - \frac{\Phi^{(M-k)}(0)}{\beta(M-k)!} - \sum_{i=1}^{k-1} (M-k+1)\cdots(M-k+i) b_{M-k+i} \frac{(\tau_T-\tau_q)^i}{i!} \tag{44}$$

and

$$C_{M-k} = a_{M-k}C_M + b_{M-k}, \quad C_0 = \Gamma(0), \\ k = 2, \ldots, M, \tag{45}$$

where

$$a_{M-1} = M\frac{1-\beta(\tau_T-\tau_q)}{\beta}, \quad b_{M-1} = -\frac{\Phi^{(M-1)}(0)}{\beta(M-1)!}.$$

Since C_0 is given, C_M can be solved from $C_0 = a_0C_M + b_0$. Once C_M is obtained, C_k can be determined based on Eq. (45). Hence, $\Gamma(t) = \sum_{i=0}^{M} C_i t^i$.

Case 2. If $\tau_T < \tau_q$, then

$$\begin{aligned} a_{M-k} = {} & \frac{1}{\beta}M(M-1)\cdots(M-k+1)\frac{(\tau_q-\tau_T)^{k-1}}{(k-1)!} \\ & + \frac{1}{\beta}\sum_{i=1}^{k-1}(M-k+1)\cdots(M-k+i) \\ & \times \frac{(\tau_q-\tau_T)^{i-1}}{(i-1)!}a_{M-k+i}, \end{aligned} \tag{46}$$

$$\begin{aligned} b_{M-k} = {} & \frac{1}{\beta}\sum_{i=1}^{k-1}(M-k+1)\cdots(M-k+i) \\ & \times \frac{(\tau_q-\tau_T)^{i-1}}{(i-1)!}a_{M-k+i} - \frac{\Phi^{(M-2)}(\tau_q-\tau_T)}{\beta(M-k)!} \end{aligned} \tag{47}$$

and

$$C_{M-k} = a_{M-k}C_M + b_{M-k}, \quad C_0 = \Gamma(0), \\ k = 1, \ldots, M. \tag{48}$$

Again, since C_0 is given, C_M can be solved from $C_0 = a_0C_M + b_0$. Once C_M is obtained, C_k can be determined based on Eq. (48). Hence, $\Gamma(t) = \sum_{i=0}^{M} C_i t^i$.

Finally, the numerical solution for Eqs. (33)–(35) can be written as follows:

$$T(x,t) = \sum_{n=1}^{\infty}\Gamma_n(t)\sin\frac{n\pi x}{l}, \tag{49}$$

where

$$\Gamma_n(t) = \sum_{i=0}^{M} C_i^n t^i. \tag{50}$$

3. Numerical examples

Two simple examples are given to test our method. The first example is a simple 1D heat conduction problem as follows:

$$\frac{\partial T(x,t+\tau_q)}{\partial t} = \frac{\partial^2 T(x,t+\tau_T)}{\partial x^2}, \tag{51}$$

$$T(x,0) = \sin\pi x \tag{52}$$

and

$$T(0,t) = T(1,t) = 0. \tag{53}$$

It can be seen that the exact solution for the no time lag case is $T(x,t) = e^{-\pi^2 t}\sin\pi x$. The numerical solution can be written as follows:

$$T(x,t) = \sum_{n=1}^{\infty}\Gamma_n(t)\sin n\pi x,$$

where

$$\Gamma_n(t) = \sum_{i=0}^{M} C_i^n t^i$$

and C_i^n is computed based on Eqs. (23) or (28). From Eq. (14), we can see that $\Gamma_1(0) = 1$ and $\Gamma_n(0) = 0$, $n = 2, 3, \ldots$ Thus, from Eqs. (23) and (28) $C_i^n = 0$, $n = 2, 3, \ldots$, and hence the numerical solution can be expressed as follows: $T(x,t) = \Gamma_1(t)\sin\pi x$.

Figs. 1–3 give the solutions for various values of τ_q and τ_T when $M = 50$ and $t = 0.01$, 0.1, and 0.5, respectively. From these figures, it is seen that for $\tau_q = \tau_T$ the exact solution and the approximate solution are the same. For $\tau_q > \tau_T$, the temperature gradient precedes the heat flux vector, implying that the temperature gradient is the cause while the heat flux vector is the effect. On the other hand, for $\tau_q < \tau_T$, the heat flux vector precedes the temperature gradient, implying that the heat flux vector is the cause while the temperature gradient is the effect. It can be seen from Figs. 1–3 that the temperature level for $\tau_q > \tau_T$ is higher than that for $\tau_q < \tau_T$. Also, the temperature level for $\tau_q > \tau_T$ goes down as τ_q increases while the temperature level for $\tau_q < \tau_T$ goes up as τ_T increases. Both levels become close to each other as τ_q and τ_T become large. Furthermore,

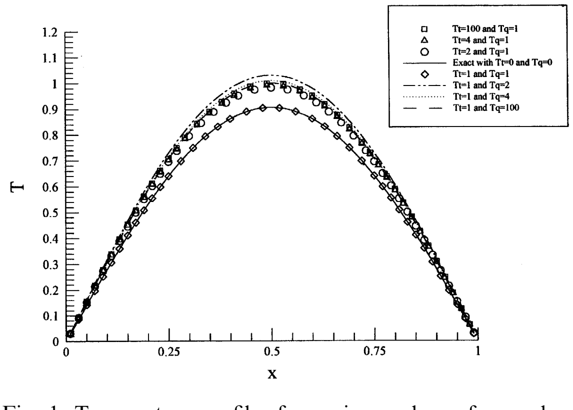

Fig. 1. Temperature profiles for various values of τ_q and τ_T when $t = 0.01$ in the first example. In the figure, *Tt* stands for τ_T and T_q for τ_q.

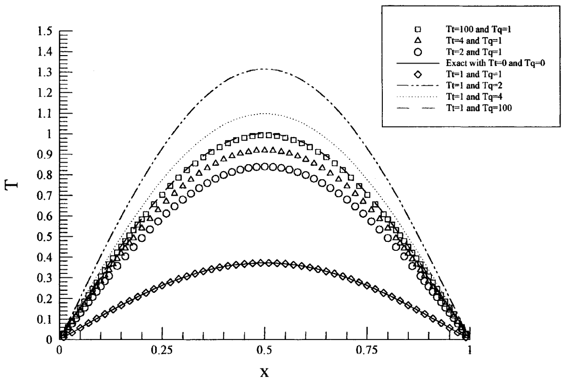

Fig. 2. Temperature profiles for various values of τ_q and τ_T when $t = 0.1$ in the first example. In the figure, Tt stands for τ_T and T_q for τ_q.

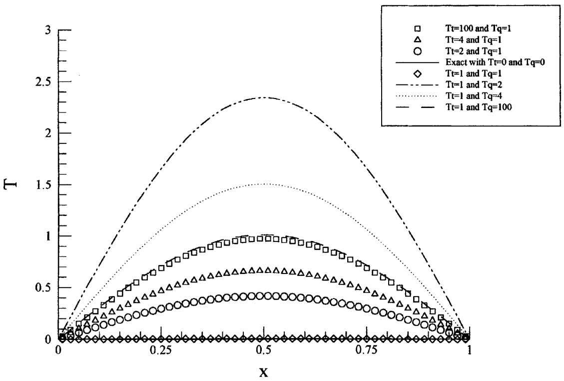

Fig. 3. Temperature profiles for various values of τ_q and τ_T when $t = 0.5$ in the first example. In the figure, Tt stands for τ_T and T_q for τ_q.

the temperature levels for both cases are higher than that predicted by the traditional heat conduction equation.

Fig. 4 shows a plot of the coefficient C_i^1 in Eq. (30) for $\tau_q = 1$ and $\tau_T = 2$ when $M = 50$, 100, and 200, which was computed using Eqs. (22)–(24). Fig. 5 shows the coefficient C_i^1 in Eq. (30) for $\tau_q = 2$ and $\tau_T = 1$ when $M = 50$, 100, and 200, which was computed using Eqs. (27) and (28). The coefficients obtained based on different M values are not significantly different in both figures. The results show that the coefficient C_i^1 is convergent.

The second example is a 1D heat conduction equation with a heat source and initial and boundary conditions as follows:

$$\frac{\partial T(x, t+\tau_q)}{\partial t} = \frac{\partial^2 T(x, t+\tau_T)}{\partial x^2} + e^{-(t+\tau_q)} \sin \pi x, \tag{54}$$

$$T(x, 0) = 0 \tag{55}$$

and

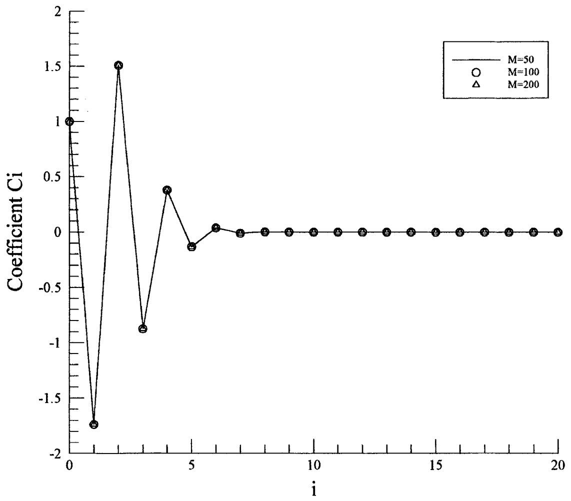

Fig. 4. Coefficient C_i for $\tau_q = 1$ and $\tau_T = 2$ when $M = 50$, 100, 200 in the first example.

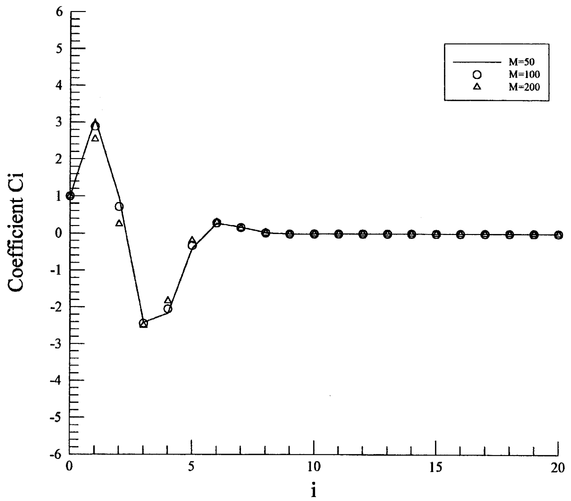

Fig. 5. Coefficient C_i for $\tau_q = 2$ and $\tau_T = 1$ when $M = 50$, 100, 200 in the first example.

$$T(0, t) = T(1, t) = 0. \tag{56}$$

It can be seen that the exact solution for the no time lag case is

$$T(x, t) = \frac{1}{\pi^2 - 1}(e^{-t} - e^{-\pi^2 t}) \sin \pi x.$$

We assume that the numerical solution can be written as follows:

$$T(x, t) = \sum_{n=1}^{\infty} \Gamma_n(t) \sin n\pi x,$$

where

$$\Gamma_n(t) = \sum_{i=0}^{M} C_i^n t^i$$

and C_i^n is computed based on Eqs. (45) or (48). From Eq. (39), it is seen that $\Gamma_n(0) = 0,\ n = 1, 2, 3, \ldots$ Expanding $S(x,t)$ in Eq. (37) in a Fourier sine series, we obtain the coefficients as follows: $\Phi_1(t) = 2\int_0^1 S(x,t) \sin \pi x \, dx = e^{-t}$, and $\Phi_n(t) = 0,\ n = 2, 3, \ldots$ Hence, the numerical solution can be expressed as follows: $T(x,t) = \Gamma_1(t) \sin \pi x$.

Figs. 6–8 give the solutions for various values of τ_q and τ_T when $M = 50$ and $t = 0.05$, 0.2, and 0.5, respectively. From these figures, it is seen that for $\tau_q = \tau_T = 0$ or $\tau_q = \tau_T = 1$ the exact solution and the numerical solution are the same. Further, the temperature level for $\tau_q < \tau_T$ is higher than that for $\tau_q > \tau_T$.

Fig. 9 shows the coefficient C_i^1 in Eq. (50) for $\tau_q = 1$ and $\tau_T = 2$ when $M = 50$, 100, and 150, respectively. The coefficient was computed using Eq. (45). Fig. 10 shows the coefficient C_i^1 in Eq. (50) for $\tau_q = 2$ and $\tau_T = 1$ when $M = 50$, 100, and 150, respectively, which was

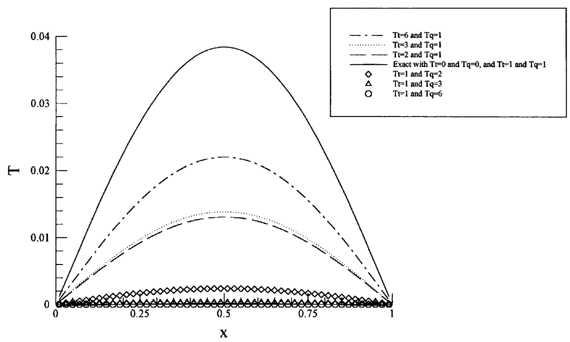

Fig. 6. Temperature profiles for various values of τ_q and τ_T when $t = 0.05$ in the second example. In the figure, *Tt* stands for τ_T and T_q for τ_q.

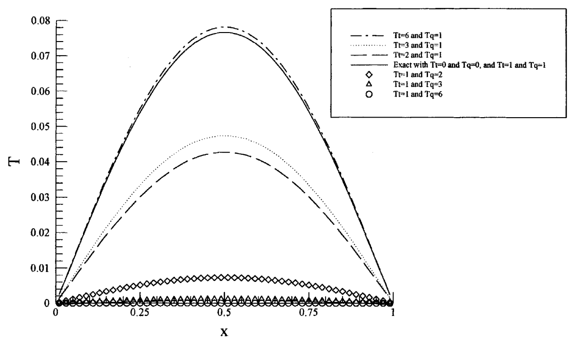

Fig. 7. Temperature profiles for various values of τ_q and τ_T when $t = 0.2$ in the second example. In the figure, *Tt* stands for τ_T and T_q for τ_q.

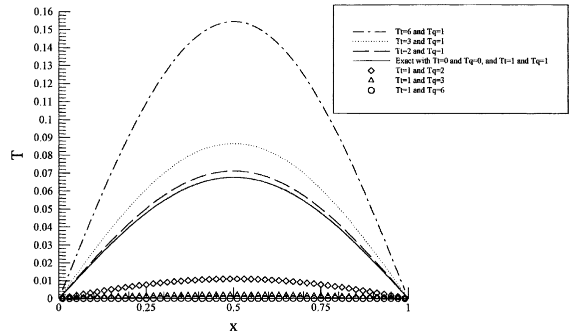

Fig. 8. Temperature profiles for various values of τ_q and τ_T when $t = 0.5$ in the second example. In the figure, *Tt* stands for τ_T and T_q for τ_q.

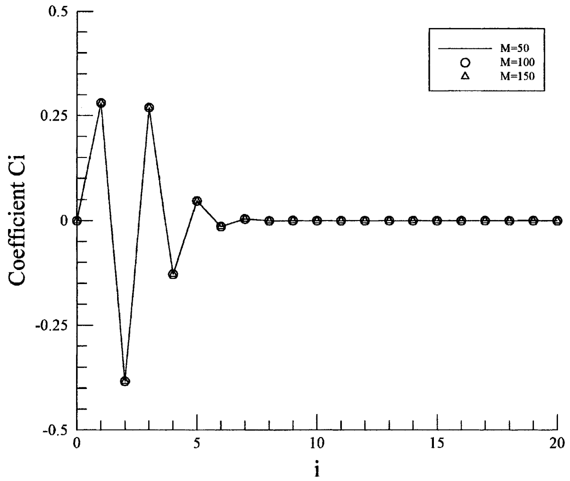

Fig. 9. Coefficient C_i for $\tau_q = 1$ and $\tau_T = 2$ when $M = 50,\ 100$, and 150 in the second example.

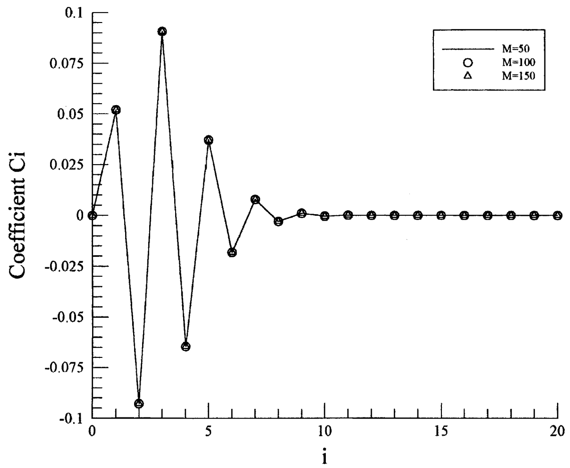

Fig. 10. Coefficient C_i for $\tau_q = 2$ and $\tau_T = 1$ when $M = 50,\ 100$, and 150 in the second example.

computed using Eq. (48). The coefficients obtained based on different M values are not significantly different in both figures. Again, the results show that the coefficient C_i^1 is convergent.

4. Conclusion

In this study, we develop a new numerical method for solving 1D dual-phase-lagging heat conduction equations. The method is illustrated with two simple examples. The method can be readily generalized to the multidimensional case because one can employ the method of separation of variables to separate variables between x, y, z and t. We will further study the application of this method to solving practical problems.

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